H. Lee - 20/10/11 - Dehn twists and exact triangles

I) Pseudoholom. sections of Lefschetz fibrations (compact base)

Def.: $\pi : E^{2n+2} \to S$ exact L. fibration.
A <u>Lagr. boundary condition</u> is $F^{n+1} \subseteq \partial^v E$ st.
$\exists\, \alpha_F \in \Omega^1(\partial S)$, $h_F \in C^\infty(F,\mathbb{R})$ st. $\theta_{E|F} = \pi_{|F}^*\underbrace{(\alpha_F + \theta_{S|\partial S})}_{\text{closed } (\Omega^1(\partial S))} + dh_F$ $\Rightarrow \theta_{E|F}$ closed

ie. • F is Lagrangian (exact if $\alpha_F + \theta_{S|\partial S}$ is)
• $\forall z \in \partial S$, F_z is exact Lagrangian, and carried into each other by parallel transport along ∂S.

Ex: (local model): $E = \{ x \in \mathbb{C}^{n+1} / Q(x) \leq r,\ |k(x)| \leq s \}$

$$\begin{array}{l} E \\ \downarrow Q : (x_1, \dots x_{n+1}) \mapsto \Sigma x_i^2 \\ D^2(r) \end{array}$$

Lagr. ∂ cond.: $F_z = \sqrt{z}\, S^n$ $\forall |z| = r$; $F = \bigcup_{|z|=r} F_z$.

Def.: <u>Relative perturbation datum</u> for $(E \xrightarrow{\pi} S, F) := (K, J)$
- $K \in \Omega^1(E)$, $K_{|TE^v} = 0$, $K_{|F} = 0 \in \Omega^1(F)$
  $K = 0$ near $\partial^h E \cup \mathrm{crit}(\pi)$
- J ω_E-compat. a.c.s. st. π is J-holom., and
  $J = I_E$ near $\partial^h E \cup \mathrm{crit}(\pi)$.

Def.: Inhomogeneous pseudohol. section w/ boundary in F:
$u : S \to E$ st. (1) $\pi(u(z)) = z$ (3) $u(\partial S) \subseteq F$
(2) $(Du - Y(u))^{0,1} = 0$, ie.
$$Du(z) + J(u) \circ Du \circ I_S = Y(u) + J(u) \circ Y(u) \circ I_S$$
where $Y \in C^\infty(E, \mathrm{Hom}(\underbrace{\pi^* TS}_{\cong\, TE^h}, TE^v))$

given $\zeta \in (TS)_z$, $z = \pi(x)$, $Y(\zeta) =$ Ham. v.f. for $K(\zeta) \in C^\infty(E_z)$

Let $\mathcal{M}_{E/S} = \{\text{such } u\text{'s}\}$; if dim. 0 , set $\Phi_{E/S} = \# \mathcal{M} \in \mathbb{Z}/2$.

Ex: in local model, pseudohol. sections (unperturbed) are

$$u_a : S \longrightarrow E \quad , \quad u_a(z) = \frac{1}{\sqrt{r}}\, a z + \sqrt{r}\, \bar{a} \quad , \quad a \in \mathbb{C}^{n+1} \iff (2)(3)$$

and $(1) \iff a_1^2 + \dots + a_{n+1}^2 = 0, \quad |a|^2 = \frac{1}{2}$

(ie. $|\mathrm{Re}\, a|^2 = |\mathrm{Im}\, a|^2 = \frac{1}{4}$, $\langle \mathrm{Re}\, a, \mathrm{Im}\, a \rangle = 0$).

$\Rightarrow \dim \mathcal{M}_{E/S} = 2n-1$, and these are regular $\rightsquigarrow$ $\Phi_{E/S} = 0$.

In fact, lemma: $E \xrightarrow{\pi} S = D^2$ with a single critical point, $F =$ vanishing cycle $\ni$ condition $\Rightarrow \Phi_{E/S} = 0$

(idea: shrinking the base disc enough ensures that sections stay within small nbd. of critical point (don't escape into other parts of fiber) so can use calc. from local model).

II) Pseudohol. sections w/ strip-like ends:

Def: LF w/ strip-like ends :

$$\begin{array}{ccc} E & \longleftarrow & Z^{\pm} \times M \\ \downarrow & & \downarrow \\ S \supset U_\zeta & \xleftarrow[\varepsilon_\zeta]{\sim} & Z^{\pm} = \mathbb{R}^{\pm} \times [0,1] \end{array}$$

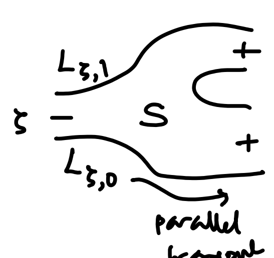
LF. is trivial over strip like ends.

S has marked pts $\Sigma \ni \zeta$ and strip like ends

(+ punctures will be inputs of operations

− —"— ———— outputs)

Lag. boundary conditions:



$F \subseteq E - \pi^{-1}(\text{marked pts})$ as before,

s.t. $\forall \zeta \in \Sigma \;\; \exists \,(L_{\zeta,0}, L_{\zeta,1})$ st.

$$F_{\varepsilon_\zeta(s,k)} = L_{\zeta,k} \quad \forall k = 0,1, \;\; \forall s \in \mathbb{R}^{\pm}.$$

* Fix F, and fix Floer datum for each $(L_{\zeta,0}, L_{\zeta,1})$.

<u>Rel. perturbation datum</u> (K,J) = as before, but on strip like ends require

$$K(s,t,x) = H_\zeta(t,x)\,dt \qquad H_\zeta \in C^\infty(M\times[0,1],\mathbb{R})$$
$$J(s,t,x) = i \oplus J_\zeta(t,x) \qquad J_\zeta \text{ t-dependent a.c.s. on } M$$

} as given by Floer data for $(L_{\zeta,0}, L_{\zeta,1})$.

$\rightsquigarrow \mathcal{M}_{E/S}(\{y_\zeta\})$ = moduli sp. of perturbed holom. sections u

$\{y_\zeta\}$, $y_\zeta \in \mathcal{C}(L_{\zeta,0}, L_{\zeta,1})$ st. $\lim_{s\to\pm\infty} u(\varepsilon_\zeta(s,\cdot)) = y_\zeta$

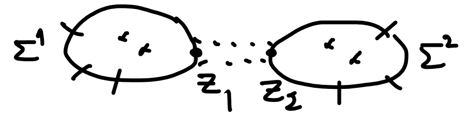
$\Rightarrow$ <u>invariant</u>: $\| \; C\Phi_{E/S} : \bigotimes_{\zeta^+\in\Sigma^+} CF(L_{\zeta^+,0}, L_{\zeta^+,1}) \longrightarrow \bigotimes_{\zeta^-\in\Sigma^-} CF(L_{\zeta^-,0}, L_{\zeta^-,1})$.

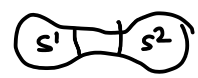
$$\bigotimes_{\zeta^+} y_{\zeta^+} \longmapsto \sum \# \mathcal{M}^0_{E/S}(\{y_{\zeta^-}, y_{\zeta^+}\}) \cdot \left(\bigotimes_{\zeta^-} y_{\zeta^-}\right).$$

(0-dim^l. only)

<u>gluing thm</u>: $E^1 \to S^1$, $E^2 \to S^2$ st. $(E^1)_{z_1} = (E^2)_{z_2} = M$

$(F^1)_{z_1} = (F^2)_{z_2} = L$



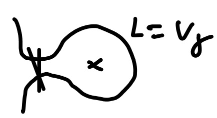
$\Rightarrow$ let $E \to S$ = gluing  w/ sufficiently long neck (small gluing parameter).

$\Rightarrow$ <u>Thm</u>: $\| \; \mathcal{M}_{E/S}(\{y_\zeta\})^0 \simeq \coprod_{p+q=n} (\mathcal{M}_{E_1/S_1})^p \times_L (\mathcal{M}_{E_2/S_2})^q$

fibered product for $ev_{z_1}: \mathcal{M}_1 \to L$
$ev_{z_2}: \mathcal{M}_2 \to L$

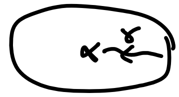
<u>Corollary</u>: $\| \;$ if $\phi_{E^2/S^2, z} = 0 \in H^*(L)$ then $\phi_{E/S} = 0$.

ev. chain at z

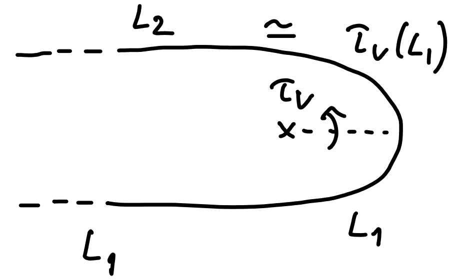
(Use this if:  $L \simeq V_\gamma \Rightarrow$ split off  & use previous lemma $\Rightarrow \phi_{E/S} = 0$).

III) • $L_0 = V$ Lagr. sphere, L_1, $L_2 \simeq \tau_V(L_1)$ in M

Build L. fibration $\pi : E \to S$ w/ one sing fiber, v. cycle $= V$



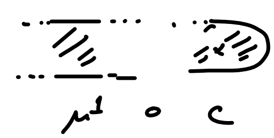
Then after punching S at one point, get:



+ fix perturb data, Floer data

$\Rightarrow$ get $c := C\Phi_{E/S} \in CF(L_1, L_2)$ satisfies $\mu^1(c) = 0$.

$\mu^1(c) = 0$ because ∂(1-dim'l spaces of sections) = broken configs

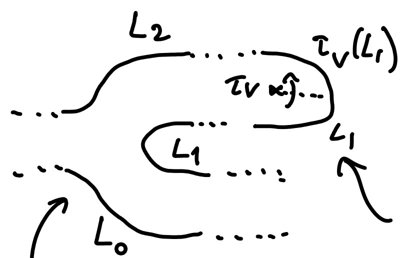


• Next, construct $k : CF(L_0, L_1) \to CF(L_0, L_2)$ s.t. $\mu^1(k(\cdot)) + k(\mu^1(\cdot)) + \mu^2(c, \cdot) = 0$

i.e. k = homotopy b/w $\mu^2(c,\cdot)$ and zero.

Done by considering a family of LF's $\begin{matrix} E^{k,r} \\ \downarrow \\ S^{k,r} \end{matrix}$ s.t. $C\Phi_{E^{k,0}/S^{k,0}} = \mu^2(c,\cdot)$, $C\Phi_{E^{k,1}/S^{k,1}} = 0$.

Namely, $\begin{matrix} E^{k,0} \\ \downarrow \\ S^{k,0} \end{matrix}$

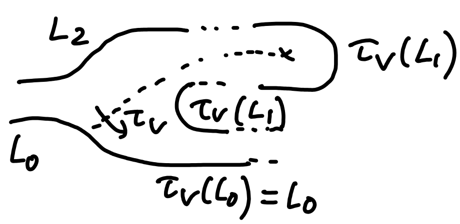


$C\phi_{E^{k,0}/S^{k,0}} = \mu^2(c,\cdot)$

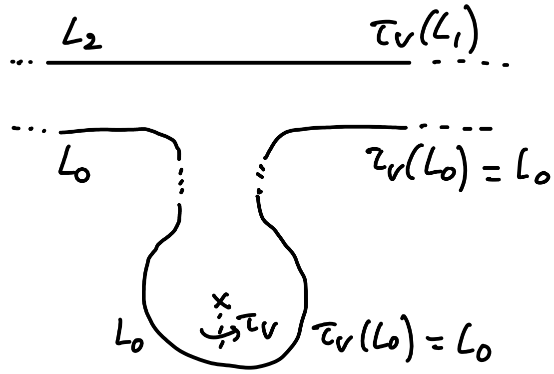
by def. $C\phi = c$.

trivial over  so $C\phi \equiv \mu^2$ Floer product

<u>Note:</u> This is the <u>same</u> as



and let $\begin{array}{c} E_{k,1} \\ \downarrow \\ S_{k,1} \end{array}$ =



$c\phi_{E_{k,1}/S_{k,1}} = 0$

by vanishing theorem.

Now, k counts exceptional (index -1) sections in the interpolating family.

Observe: $\bigsqcup_{r\in[0,1]} \mathcal{M}^0(E^{k,r}/S^{k,r})$ is a 1-mfld with boundary =

- $\mathcal{M}^0(E^{k,0}/S^{k,0})$ gives $\mu^2(c,\cdot)$
- $\mathcal{M}^0(E^{k,1}/S^{k,1})$ gives 0
- broken configs. $\coprod_{r\in(0,1)} \mathcal{M}^{-1}(E^{k,r}/S^{k,r}) \times (\text{strips})$

  gives $\mu^1(k(\cdot)) \pm k(\mu^1(\cdot))$.